Yang-Mills Theory at 60:
Milestones, Landmarks, and Interesting Questions

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On the auspicious occasion of celebrating the 60th anniversary of the Yang-Mills theory, and Professor Yang's many other important contributions to physics and mathematics, I will highlight the impressive milestones and landmarks that have been established in the last 60 years, as well as some interesting questions that still lie before us. The paper is written (without equations) for the interest of non-scientists as well as of scientists.

Keywords: Yang-Mills Theory at 60 [a]

CONTENTS

[a] Invited contribution to the Conference on 60 Years of Yang-Mills Gauge Theories, IAS, NTU, Singapore, 25-28 May 2015; to appear in the Proceedings and in IJMPA (International Journal of Modern Physics A) Vol. 30 (2015) 1530068 (27 pages),© World Scientific Publishing Company
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1. Overview [b]

The 1954 Yang-Mills (YM) paper,[1] that set forth the YM theory with YM equations, made at least the following three major advances, if not revolutions, in theoretical and mathematical physics.

First, the YM equations generalized the Maxwell equations[2] from the Abelian gauge field equations, in which the Maxwell fields do not interact with themselves, to the more general non-Abelian gauge field equations, in which the YM fields do interact with themselves. It is akin to Einstein's general relativity (GR) equations[3] generalizing Newton's equations for gravity,[4] though in different ways. The Maxwell equations are exact by themselves and the YM equations are generalizations, leaping conceptually from the simpler special case of the Maxwell equations. In contrast, Newton's equations for gravity are weak-field approximations of Einstein's GR equations, so they still work for our daily lives and for the motions of the planets, except for minor but measurable GR improvements to agree with Nature, e.g., in the workings of the global positioning system (GPS) and in the precession of the perihelion of Mercury.

Second, the YM equations (including the anti-self-dual and the self-dual YM equations) are beautiful nonlinear partial differential equations for physics and have inspired the development of beautiful mathematics, as has been the case with Einstein's GR equations.

Third, the YM theory manifested the local gauge covariance, which was first established in Maxwell's equations for electromagnetic interactions, as a principle for more general interactions (later found to be weak and strong interactions, see below). In contrast, Einstein's GR theory manifests the local coordinate covariance as the principle for gravitational interactions.

After decades of further theoretical development and experimental work, amazingly, Nature is found to make use of these theoretical advances for interactions among elementary matter fields.[c] The SU(3) Yang-Mills fields are the mediating force fields for strong interactions. The SU(2) Yang-Mills fields are the mediating force fields for weak interactions, unifying in an intricate way with the U(1) Maxwell fields to become the U(1) x SU(2) Maxwell-Yang-Mills (MYM) mediating force fields for the unified electroweak interactions. Nature surprisingly chooses to use the simplest types of groups,[d] U(1) x SU(2) x SU(3), for electroweak and strong interactions, three of the four major interactions known in Nature.

For the last thirty years of his life, Einstein had searched for, unsuccessfully, the answer to the question:[5] Does Nature have a unified theory for all interactions? As the end of 2015 is approaching, sixty years after his passing in 1955, the answer to his question is still yet to be found. However, he

[b] I would recommend that readers first read sections I. (Overview) and IV. (Concluding Remarks), before reading sections II. and III., which cover a vast amount of information in highlight fashion and are supplemented with references and ample sources for references.

[c] The experimentally established elementary matter fields, as of 2015, are the fields of quarks, leptons, and the Higgs boson of mass 125Gev/c^2.

[d] A group is a set of mathematical elements, together with an operation called group-multiplication, under which the group has an identity and every group element has its inverse. Groups are the second simplest mathematical structure, next to sets, yet are so prominently used by Nature in formulating laws of interactions.

might be pleased to know the progress mankind has made toward answering his question. At the time of his passing, when Yang-Mills theory was less than one year old, among the four experimentally established major interactions: electromagnetic, weak, strong, and gravity, only for electromagnetism had a quantum field theory been established: quantum electrodynamics (QED), with the quantum Maxwell fields being the U(1) “glues”. Now a quantum field theory has been established for electroweak interactions, quantum flavor dynamics (QFD), which unifies electromagnetic and weak interactions in an intricate way. The quantum Maxwell fields remain to be the U(1) “glues” as in QED and the quantum Yang-Mills fields are the SU(2) “glues”. Additionally, a quantum field theory has been established for strong interactions, quantum chromodynamics (QCD), in which the quantum Yang-Mills fields are the SU(3) “glues”. So Einstein’s question can be asked in more specific terms: Will we find that QFD and QCD (the combination of the two is called the Standard Model of particle physics) are parts of a grand unified theory (GUT)? Will we find the ultimate quantum description of Einstein’s general relativity (QGR)? Will we find the ultimate unified theory, The Unified Theory (TUT) or The Theory of Everything (TOE), of GUT and QGR? To know how these questions are answered will make life richer.[e]

2. Yang-Mills theory: milestones and interesting questions

The milestones to be highlighted in this section, except for Milestone II.4., have all been theoretically developed and experimentally established, and set forth in modern graduate textbooks on quantum field theories. See the authoritative volumes with comprehensive and exhaustive references by S. Weinberg,[6] and the celebration volume for the 50th anniversary of the Yang-Mills theory edited by ‘t Hooft,[7] for which Attachment A provides a copy of its title page, Preface and Contents. Therefore, l will be very brief in highlighting the milestones in this section without giving references (except a few) to original papers which Refs. [6,7] comprehensively and exhaustively have provided.

It is interesting to note that among the sixty-one Nobel Prizes in Physics[8] from 1954 to 2014, twenty-six are related to the making of the Standard Model, while the 2015 one honors the experiment that gave evidence for the need of going beyond the Standard Model (see Milestone 5).

Milestone 1. Quantum flavor dynamics QFD has been theoretically developed and experimentally established for electroweak interactions,[f] unifying electromagnetic and weak interactions. In QFD the quantum SU(2) Yang-Mills fields are the quantum mediating force fields for weak interactions, i.e., the SU(2) “weak glues”, among elementary particles with SU(2) “weak charges”, while the quantum U(1) Maxwell fields remain, as in QED, to be the quantum U(1) “electromagnetic glues” among particles with U(1) “electromagnetic charges”. In this intricate quantum unification the Maxwell field quanta, photons or light, are massless (as in QED) and can travel large distance in space void of matter with electromagnetic properties. Thus we can see each other, the moon, the sun, stars and galaxies. On the other hand, the SU (2) Yang-Mills quanta gain masses, through the Higgs mechanism (see Milestone 4. below), to become the massive W^+, W^- and Z^0 particles that can travel only very short distances.

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[e] Nature has been kind to humans in allowing their brains to understand the complexity of Nature bit by bit over time and then bring the pieces of understanding together into larger structures of meaning. How long will this process continue? Will the human brain ever discover the totality of Nature (finite or infinite)?

[f] The letter F in QFD is to indicate the fact that quarks and leptons exist in three different types called “flavors”.

Interestingly, among the four major interactions, **only** weak interactions, mediated by the SU(2) YM quanta, W^+, W^- and Z^0, violate the discrete symmetries P (parity) and CP (parity and charge conjugation), as well as T (time reversal), while CPT symmetry stays true in all interactions.

Milestone 2. Quantum chromodynamics QCD has been theoretically developed and experimentally established for strong interactions.[g] In QCD the quantum SU(3) Yang-Mills fields are the quantum mediating forces for strong interactions, the SU(3) "color glues" of quanta called gluons. They interact with quarks according to their strong-interaction charge, the SU(3) "color charge". (The leptons and the Higgs particle are SU(3) "colorless", and therefore do not participate in strong interactions).

The QFD quanta with the U(1) and the SU(2) electroweak charges and "flavors" can exist freely. In contrast, the QCD quanta with non-zero SU(3) "color charges" cannot exist freely. Or, they are "color confined" or "color enslaved". They have to combine to form SU(3) "colorless" particles to exist freely. For example, protons and neutrons are SU(3) "colorless" particles formed from "colorful" quarks and gluons. The SU(3) "colorless" particles formed from "colorful" gluons are called glueballs, which are still under theoretical study[9] and experimental search.[10]

Also unusual is that the binding forces from the gluons decrease as distances decrease (corresponding to increasing energies) between the quarks --- so QCD is an asymptotically free theory, quite opposite to that of QED.[11] These make QCD manageable to calculate in high energies and in high temperatures[12] This has implications for cosmological studies on the early universe and for high energy experiments.[13]

Milestone 3. Local gauge covariance has been established to be the underlying principle of how Maxwell and Yang-Mills fields act as "glues" among the basic matter fields[14,15] --- the Maxwell fields as the "electromagnetic glues" and the Yang-Mills fields as the "weak glues" as well as the "strong (color) glues". This principle has been used to develop new theories, e.g., the supersymmetric Yang-Mills theories, SYM,[16] and even for supersymmetric gravity, SGR.[17]

Milestone 4. A Higgs particle (possibly the first of many) of the Higgs mechanism (the ubiquitous "molasses"),[18] which produces the masses of quarks and the masses of W^+, W^- and Z^0 (the SU(2) Yang-mills quanta in weak interactions), was finally observed in 2012,[19,20] almost fifty years after the original theoretical proposals. It was an admirable triumph of the collaborative spirit in scientific research!

Milestone 5. Toward the end of the twentieth century, experiments established that the mass differences between any two of the three-flavored neutrinos are not zero, in contradiction to the all-

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[g] The letter C in QCD is to indicate the fact that the "SU(3) gluons" have different SU(3) quantum identities called SU(3) "colors", conjured up by the color of photons, the "U(1) gluons".

zero-mass neutrinos in QFD of the Standard Model. This fact forces consideration of going beyond the Standard Model.[21,22, 23]

Interesting questions

- As highlighted above, Nature has made versatile uses of the Yang-Mills fields, but not for uses on the macroscopic scales like the Maxwell fields and the Newton-Einstein fields.[24] Can a precise mathematical physics explanation for this be found? [25]
- Is there a deeper guiding principle for the Higgs mechanism, and how many Higgs particles there are? Will the mass matrices of quarks and of leptons (which are generated by the presence of the Higgs fields) be derived, so hopefully a deeper understanding of the origin of CP violations in weak interactions will also follow?
- What will be established to be the Beyond Standard Model? Will the supersymmetric extensions[26] become experimentally observed realities?
- Will a grand unification theory, GUT, be found in which the electroweak and the strong interactions are unified?
- Will GR be established theoretically and experimentally as a quantum theory?

For a succinct and insightful perspective on the last three questions, see Witten.[27]

3. The (Anti-)Self-dual Yang-Mills Fields: landmarks and interesting questions

The Yang–Mills fields in even dimensions of spacetime can be expressed as a sum of two terms: the self-dual term and the anti-self-dual term. Fields that are self-dual (SDYM) or anti-self-dual (ASDYM) are special because they automatically satisfy the Yang-Mills equations, and in addition, by definition, satisfy non-linear differential equations, one-order lower than the Yang-Mills equations, which are called the (A)SDYM equations. (A)SDYM fields are simpler to study, while still remaining interesting and important for physics and mathematical physics research.

Landmark 1. The (A)SDYM fields have interesting solutions called instantons, which possess specific topological members (the Pontryagin or the second-Chern number) and minimize the Yang-Mills action (functional).

In 1975, Belavin, Polyakov, Schwartz and Tyupkin constructed explicitly the first such solutions,[28] which stimulated much interest in mathematics as well as in physics.

In 1977 Ward showed[29] that (A)SDYM fields are naturally described by the twistor formulation that Penrose developed[30] to describe massless fields and the self-dual Einstein fields, and derived what are now called the Penrose-Ward transformations. Not long afterwards, this breakthrough by Ward led to the full description of the space of instantons in 1979 by Atiyah, Drinfeld, Hitchin and Mannin (now called the ADHM construction).[31] See the comprehensive review lecture by Atiyah and references to original papers.[32]

The instanton solutions prompted intensive research on their non-perturbative effects in QCD and QFD,[33] and led to experimentally testable predictions, for example the axions[34] and the related sphaleron phenomena.[35]

Landmark 2. For (A)SDYM fields in four-dimensions Yang showed[36] that (A)SDYM fields can be written as second-order nonlinear partial differential equations, now called the Yang equations, in terms of group-value local fields.[h]

Later the Yang equations were found to have many characteristics of the classical integrable systems in lower dimensions: Backlund transformations,[37] non-local conservation laws,[38] Riemann-Hilbert transformation properties,[39] Painleve properties,[40] generalized Riccarti equations,[41] and, most importantly, the linear systems (the Lax systems),[42] and Kac-Moody algebras.[43,44,45,46] For an overview of the work and references during this period of discoveries, see <*Integrable Systems*>,[47] the lectures therein[48,49] and Attachment B below, which gives the memorable front pages of the book. For a glimpse of the landscape of mathematical physics in the late 1980s, see Chau and Nahm.[50]

Additionally, many integrable systems in lower dimensions, including the famed Nahm equations,[51] can be directly reduced from the Yang equations. See the in-depth reviews[52,53] and the extensive references therein. See the overview[54] of (A)SDYM as a classical integrable system in four dimensions and its relations to those in lower and higher than four dimensions. The review Ref. [48] includes discussions on extensions to supersymmetric theories, SYM and SGR, which will be discussed in Landmark III.4.

Therefore, (A)SDYM equations have been shown to be classical integrable systems in **four** dimensions that interestingly also serve as conceptual pathways between those integrable systems in dimensions lower and higher than four dimensions, as well as in supersymmetric dimensions.

The next important step was to develop (A)SDYM equations into a quantum integrable field theory in four dimensions. In the 1990s, the Yang equations were put to quantization, using the action (functional) constructed by Nair and Schiff[55] and by Hou and Song,[56] in terms of the Lie-valued fields[55,57] and in terms of the group-valued local fields.[58] In Ref. [58] Yamanaka and I found that the quantized group-valued local fields are bi-module fields which satisfy intricate exchange algebras with structure coefficients satisfying generalized Yang-Baxter equations. So here Yang-Mills met Yang-Baxter for the first time with the quantum group properties,[59,60] originated from the Yang-Baxter equations in low dimensions, appearing in this four-dimensional quantum theory for the first time. Further, the algebraic relations satisfied by the Lie-valued fields derived in Refs. [55,57] can be derived from the exchange algebras of the group-value local fields. For further developments in the 2000s, see Popov-Preitschopf[61] and Popov.[62]

However, the quantum contents of the Yang equations have not been fully developed in detail, and the full quantum versions of its classical integrability properties are yet to be revealed.

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[h] Group-valued local fields are special. Usually group-valued fields are non-local, defined on a loop, not at points as local fields are.

Landmark 3. Using (A)SDYM fields as tools, Donaldson[63,64] made major discoveries about the four-manifolds.[i]

Here was what Atiyah said on the occasion of the awarding of the 1986 Fields Medal to Donaldson,[65] referring to Donaldson's paper Ref. [63] as [1]:

"In 1982, when he was a second-year graduate student, Simon Donaldson proved a result [1] that stunned the mathematical world. Together with the important work of Michael Freedman (described by John Milnor), Donaldson's result implied that there are "exotic" 4-spaces, i.e., 4-dimensional differentiable manifolds which are topologically but not differentiably equivalent to the standard Euclidean 4-space R4. What makes this result so surprising is that n = 4 is the only value for which such exotic n-spaces exist. These exotic 4-spaces have the remarkable property that (unlike R4) they contain compact sets which cannot be contained inside any differentiably embedded 3-sphere!

To put this into historical perspective, let me remind you that in 1958 Milnor discovered exotic 7-spheres, and that in the 1960s the structure of differentiable manifolds of dimension > 5 was actively developed by Milnor, Smale (both Fields Medalists), and others, to give a very satisfactory theory. Dimension 2 (Riemann surfaces) was classical, so this left dimensions 3 and 4 to be explored. At the last Congress, in Warsaw, Thurston received a Fields Medal for his remarkable results on 3-manifolds, and now at this Congress we reach 4-manifolds. I should emphasize that the stories in dimensions 3, 4, and n > 5 are totally different, with the low-dimensional cases being much more subtle and intricate.

The surprise produced by Donaldson's result was accentuated by the fact that his methods were completely new and were borrowed from theoretical physics, in the form of the Yang-Mills equations. These equations are essentially a nonlinear generalization of Maxwell's equations for electro-magnetism, and they are the variational equations associated with a natural geometric functional. Differential geometers study connections and curvature in fibre bundles, and the Yang-Mills functional is just the L2-norm of the curvature. If the group of the fibre bundle is the circle, we get back the linear Maxwell theory, but for nonabelian Lie groups, we get a nonlinear theory. Donaldson uses only the simplest nonabelian group, namely SU(2), although in principle other groups can and will perhaps be used.

Physicists are interested in these equations over Minkowski space-time, where they are hyperbolic, and also over Euclidean 4-space, where they are elliptic. In the Euclidean case, solutions giving the absolute minimum (for given boundary conditions at ∞) are of special interest, and they are called instantons.

Several mathematicians (including myself) worked on instantons and felt very pleased that they were able to assist physics in this way. Donaldson, on the other hand, conceived the daring idea of reversing this process and of using instantons on a general 4-manifold as a new geometrical tool. In this he has been brilliantly successful: he has unearthed totally new phenomena and simultaneously demonstrated that the Yang-Mills equations are beautifully adapted to studying and exploring this new field.

Donaldson's works continue to be highly honored.[66]

Then in 1994 Witten[67] showed that the Seiberg-Witten equations[68] of N=2 SYM also provide a powerful tool for analyzing four-manifolds, confirming those by Donaldson and discovering new properties, which generated much excitement among mathematicians.[69]

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[i] A manifold is a space that can be covered by local patches that are like the Euclidean space, or the Minkowski space.

Landmark 4. The landmarks above resulting from studying the (A)SDYM fields have inspired extensions into supersymmetric theories. In 1978, Witten generalized the concept advanced by Ward in Landmark 1,[70] that (A)SDYM equations are the results of certain integrability conditions,[29] to YM and SYM. Of particular interest is that the integrability along light-like lines precisely gives the N=3 SYM equations.[16] This naturally led to the discovery of many classical integrability properties of the theory and also to the construction of group-valued local fields and the derivation of their equations, giving the generalized Yang equations of SYM.[71]

In 1986 Lim and I showed that N=5,6,7,8 SGR, in the linearized approximation, had a similar interpretation.[72]

In 1992, I derived the linear systems for high-N SGR,[73] which signal the classical integrability properties, and constructed group-valued local fields and derived the equations they satisfy, giving the generalized Yang equations of SGR.

Recently, quantum N=4 SYM and its quantum integrability properties have been studied very actively,[74] giving a renaissance to studies on S-matrices.[75,76,] For the current status, see the paper by Brink in this Proceedings.[77]

So the landmarks of this section have revealed that (A)SDYM fields in four dimensions are important to study,[j] both for physics and for mathematical physics. They also serve importantly as conceptual pathways between integrable systems in lower and higher dimensions, as well as to those in supersymmetric spaces.

Interesting questions

- What will be the experimental observations of the effects related to instantons?
- What will be the quantum manifestations of the classical integrality properties of the Yang equations? Do quantum bi-module fields have physical reality?
- Will experiments find that Nature uses supersymmetric theories, SYM and SGR?
  If yes, such a huge discovery will inevitably lead us to ask whether SYM or SGR theories have quantum integrability properties?
- What are the implications to physics of the four-manifolds discovered by Donaldson?

[j] In Attachment C, I propose to add the topic of (A)SDYM fields to the list of the other thirteen topics that Professor Yang considered important and to which he made major contributions.

4. Concluding Remarks

In the 60 years since the Yang-Mills paper, we have seen the establishment of quantum flavor dynamics, QFD, unifying electromagnetic and weak interactions into electroweak interactions, with Maxwell fields acting as the “U(1) glues” and Yang-Mills fields acting as the “SU(2) glues”. The quanta of the U(1) Maxwell fields, photons, are massless. They can travel far and are our light (in its full spectrum). In contrast, the quanta of SU(2) Yang-Mills fields for weak interactions gain masses from the Higgs mechanism and become massive W^+, W^- and Z^0 particles. They can travel only for very short distances.

We have also seen the development of quantum chromodynamics QCD for strong interactions, with the SU(3) Yang-Mills fields acting as the “SU(3) color glues” and its quanta, the massless gluons, interacting among themselves and mediating interactions among quarks that have “SU(3) color charges”. Unlike particles with U(1) electric charges and/or SU(2) weak-interaction charges, the “SU(3) colored” gluons and quarks cannot exist freely. To be free they must form “SU(3) colorless” particles: glueballs made of gluons, hadrons (protons, neutrons, etc.) made of quarks and gluons, or be at super high energies or super high temperatures.

Nature chooses these versatile ways to use the Yang-Mills fields; however, it does not allow us to experience the Yang–Mills fields in our daily lives as we do the Maxwell fields and the Newton-Einstein fields. Why this is so is an interesting mathematical physics question worthy of an answer.

After the celebration of 60 years of the Yang-Mills Theory in 2014-2015 comes the celebration of 100 years of Einstein’s theory of general relativity in 2015-2016. We are now being treated to progress reports about the advances made in our understanding it,[78,79] in particular its quantum theoretical description and experimental attempts to observe its possible quantum phenomena. As of now, near the end of 2015, no definitive quantum theoretical description for general relativity has been established and no quantum phenomena of general relativity have yet been observed --- no gravitational waves, not to mention the more illusive gravitons, their quanta.

The ultimate question is whether Nature makes use of a unified theory for all interactions.

With all these interesting questions, mankind will be kept busy intellectually for generations to come. We look forward to seeing what new discoveries will be celebrated when Yang-Mills theory turns 70 in 2024 and Einstein’s general relativity turns 110 in 2025.[80]

Acknowledgements

I would like to give my heartfelt thanks to many friends, especially Karen Andrews, Nicolai (Kolya) Reshetikhin, Gloria Rogers, William (Bill) Rogers, and Weiben Wang, for reading the drafts and making helpful comments which improved the presentation of the paper, and for their warm friendship and encouragement that made the task of writing this paper more enjoyable; and offer my deep gratitude to Kolya, Professor of Mathematics, and Bill, Professor of English, for their expert critiques that sharpened the paper. Of course, any errors in the paper are mine alone.

Attachment A

The Title Page, Preface, and Contents of <*Yang-Mills at 50*>, Ed. G. 't Hooft (World Scientific, 2004)

50 Years of
Yang–Mills
THEORY

edited by Gerardus 't Hooft
Utrecht University, The Netherlands

World Scientific
NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI · CHENNAI

PREFACE

At the time of the publication of this volume, fifty years have passed since the appearance of an article in *The Physical Review* by Chen Ning Yang and Robert L. Mills, entitled "Conservation of Isotopic Spin and Isotopic Gauge Invariance". This book on the one hand serves as a tribute to that monumental piece of work, and on the other intends to show how its subject has evolved since that time, highlighting the landmarks that followed after the original paper emerged, and allowing its authors to indulge in new ideas and concepts. Gauge Theory has indeed grown into a pivotal concept in the Theory of Elementary Particles, and it is expected to play an equally essential role in even more basic theoretical constructions that are speculated upon today, with the aim of providing an all-embracing picture of the universal Laws of Physics.

Some of the chapters in this book are contributions that have appeared elsewhere; most of the contributions are original pieces of work. All are accompanied by brief comments by the Editor. Needless to state that this volume is far from complete. There are numerous well-known landmarks that we could not cover. Furthermore, like most developments in Science, progress not only comes from the relatively small set of papers by famous authors that enjoy enormous scores on citation indices, but it predominantly comes from the large crowds of scientists who confirm and reproduce the original research while adding inconspicuous but essential bits of understanding, not only by writing papers, but also by lecturing to students, by performing experiments and doing calculations. Without them, this book could not have been written.

Attachment A (Continued)

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Attachment A (Continued)

Attachment A (Continued)

Attachment B

The Title page, Forwards by S.S. Chern and C.N. Yang, and Contents of <*Integrable Systems*>, Nankai University, Tianjin, China; Ed. X.C. Song, (World Scientific, 1987)

Nankai Lectures on Mathematical Physics

INTEGRABLE SYSTEMS

Nankai Institute of Mathematics, China
August 1987

Editor:
Song Xing-Chang
Peking University

World Scientific
Singapore • New Jersey • London • Hong Kong

Attachment B (Continued)

数學物理

永結一体

陳省身

一九八八年九月

Mathematics and Physics
One Body Forever

S.S.Chern

Attachment B (Continued)

数学物理是很古老的学科·近年来发展尤其迅速·南开数学所理论物理研究室将出版一系列关於此学科的书籍,这是很有意义的事·

楊振宁

Mathematical physics is a subject of study which dated far back to ancient times, and has found speedy development in recent years.

It is most significant that the Theoretical Physics Division of Nankai Institute of Mathematics will publish a series of books on the subject.

C.N.Yang

Attachment B (Continued)

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Attachment B (Continued)

Attachment B (Continued)

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WORKSHOP ON GEOMETRICAL INTEGRABILITY AND EQUATIONS OF MOTION IN PHYSICS

	Aug. 18	Aug. 19	Aug. 20	Aug. 21	Aug. 22
8:30-9:30	Chau	Chau	Chau	Chau	Chau
10:00-11:00	"	"	"	"	"
Lunch					
2:30-3:30	Song	Yan	Ni	Contribu-	Panel
4:00-5:00	Hou	Ge		tions	Discussion
				"	

Topics:

L.-L. Chau (乔玲丽), University of California, Davis

I. Generic properties of geometrically integrable systems: Flow Chart, and two dimensional systems;

II. Self-dual Yang-Mills Equations;

III. Supersymmetric Yang-Mills systems;

IV. Supergravity;

V. Summary and outlook.

M.-L. Ge (葛墨林), Lanchou University and Nankai University, Tianjing
Integrability of Belinski-Zakharov Gravity and Solitons in Gravitational Wave.

B.-Y. Hou (侯伯宇), Xibei University, Xian
Virasoro and Kac-Moody of Ernst equation.

G.-J. Ni (倪光炯), Fudan University, Shanghai
A general method of calculation of anomalies.

X.-C. Song (宋行長), Beijing University, Beijing
Integrability properties of Ernst equation.

M.-L. Yan (闫沐霖), Science and Technology University, Hofei
Potts Model, Yang-Baxter Equation's Solutions; Surfaces with Genus > 1.

Attachment C

C.N. YANG's COMMANDMENTS in PHYSICS

When celebrating Professor Yang's 90th birthday, 2012, Tsinghua University presented him a black rock cube with his thirteen important contributions to physics listed and carved on the vertical faces, grouped into four major categories in physics (one on each vertical face of the cube), which I list in the table below (the first thirteen in black letters).

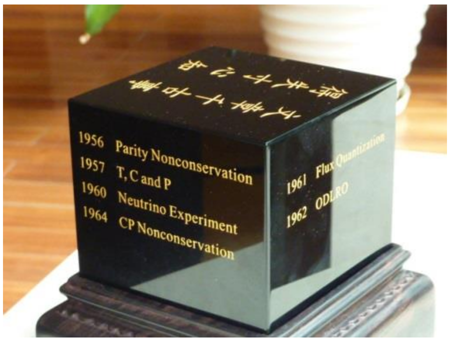
(The two photos of the cube below are copies of [Ph 72] and [Ph 73] in C.N. Yang, <*Selected Papers II with Commentaries*>, World Scientific, 2013. In the table below, I adopt the same numbering system and English translation as given by Yu Shi, *Beauty and physics: 13 important contributions of Chen Ning Yang,* Intl. J. of Mod. Phys. A29 (2014) 1475001.)

Face-A: Statistical Mechanics: A.1. 1952 Phase Transition A.2. 1957 Bosons A.3. 1967 Yang–Baxter Equation A.4. 1969 Finite Temperature	Face-B. Condensed Matter Physics: B.1. 1961 Flux Quantization B.2. 1962 ODLRO
Face-C. Particle Physics: C.1. 1956 Parity Nonconservation C.2. 1957 T, C and P C.3. 1960 Neutrino Experiment C.4. 1964 CP Nonconservation	Face-D. Field Theory: D.1. 1954 Gauge Theory D.2. 1974 Integral Formalism D.3. 1975 Fiber Bundle. D.4. 1977 (Anti-)Self-dual Gauge Fields

Here I would propose to call the listed important contributions

C.N. YANG's COMMANDMENTS in PHYSICS

and to add "1977 (Anti-)Self-dual Gauge Fields" to the list on Face-D (as D.4., shown in red) so to become the Fourteenth Commandment.



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[1] C.N. Yang and R. Mills, *Isotopic Spin Conservation and a Generalized Gauge Invariance*, October 1, Phys. Rev. 95, 631 (1954)

[2] J.M. Maxwell, *A dynamical theory of the electromagnetic field*, (PDF), Philosophical Transactions of the Royal Society of London 155: 459–512, (1865), doi:10.1098/rstl.1865.000. (This article followed a Dec. 8, 1864 presentation by Maxwell to the Royal Society.) So, Maxwell Theory turned 150 in 20l4 while YM Theory 60.

[3] A. Einstein, *Feldgleichungen der Gravitation* (English translation: *The Field Equations of Gravitation*) Preussische Akademie der Wissenschaften, Sitzungsberichte, 844–847, part 2, (1915); according to A. Pais of reference below (Chap. 14, ref. E1; Chap. 15, ref. E15), this is the defining paper of general relativity (GR) after several earlier publications on GR in 1915, which was then followed by many more in the following years, see https://en.wikipedia.org/wiki/List_of_scientific_publications_by_Albert_Einstein#Journal_articles . So, Einstein's GR turns 100 in 2015, while Newton's Principia turns 328, see Ref. [4].

[4] I. Newton, <*Philosophiæ Naturalis Principia Mathematica*>, three books in Latin, first published 5 July 1687, often referred to as <*the Principia*>; English translation <*The Mathematical Principles of Natural Philosophy*>, 1729.

[5] A. Paris, <*Subtle is the Lord: The Science and the Life of Albert Einstein*>, Oxford University Press, 1982.

[6] S. Weinberg, <*The Quantum Theory of Fields*>: Vol.I, *Foundations*, Vol.II, *Modern Applications*, and Vol.III, *Supersymmetry*, (Cambridge University Press, respectively 1995, 1996, 2000)

[7] G. 't Hooft (editor), <*50 Years of Yang-Mills Theory*>, (World Scientific, Singapore, 2005)

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[13] CMS Collaboration (V. Khachatryan et al.), *Evidence for Collective Multiparticle Correlations in p-Pb Collisions*, Phys. Rev. Lett. 115, 012301 (2015); and news http://earthsky.org/human-world/lhc-creates-liquid-from-big-bang
LHC creates liquid from Big Bang. earthsky.org/human-world SEP 15, 2015.

[14] C.N. Yang, *The Conceptual Origins of Maxwell's Equations and Gauge Theory,* Physics Today 67; 45 (2014)

[15]C.N. Yang, *Gauge Invariance and Interactions*, in Ref. [7]; It was a copy of his notes hand-written in 1947 when he was a graduate at University of Chicago --- impressive that a graduate student would think about such a big question! A typed transcript of the notes is given in Attachment-1 of L.L.

Chau's "*Almost Everything Yang Touched Eventually Turned into Gold*", Proceedings of the Conference in Honor of CN Yang's 85th Birthday, Editors: C. H. Oh, K. K. Phua, N. P. Chang and L. C. Kwek (World Scientific 2008),
http://chau.physics.ucdavis.edu/Yang85-byChau-fromWS081104f.pdf ,
Also Attachment-2 of this Chau's paper gives an anecdote about Wigner's attempt to understand gauge potential.

[16] L. Brink, J.H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills Theories,* Nucl. Phys. B 121, 77(1977); G. Gliozzi, J. Scherk and D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl. Phys. B 122 (1977) 253; and references therein for earlier papers.

[17] P. van Nieuwenhuizen, *Superg,ravity as a Yang-Mills Theory*, in this Proceedings; D.Z. Freedman and A. Van Proeyen, <*Supergravity*>, (Cambridge 2012)

[18] P.W. Higgs, *Spontaneous Symmetry Breakdown without Massless Bosons,* Phys. Rev. 145, 1156 (1966); F. Englert, R. Brout and M.F. Thiry, *Vector mesons in presence of broken symmetry,* Nuovo Cim. A 43, no. 2, 244 (1966); For an overview and more references, see
http://www.nobelprize.org/nobel_prizes/physics/laureates/2013/advanced-physicsprize2013.pdf .

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[20] CMS Collaboration, *Observation of a new boson at a mass of* 125 GeV *with the* CMS *experiment at the* LHC, Phys. Lett. B716, 30 (2012).

[21] Particle Data Group (S. Raby), *Grand Unified Theories*, (2011)
http://pdg.lbl.gov/2014/reviews/rpp2014-rev-guts.pdf ;
See also Wikipidia, *Physics Beyond the Standard Model*,
https://en.wikipedia.org/wiki/Physics_beyond_the_Standard_Model

[22] The leaders of the experiments are honored by the 2015 Nobel Prize in Physics,
http://www.nobelprize.org/nobel_prizes/physics/laureates/2015/
which gives an overview and references about neutrino oscillation at
https://www.nobelprize.org/nobel_prizes/physics/laureates/2015/advanced-physicsprize2015.pdf ;
They are also honored by the Breakthrough Prize,
https://breakthroughprize.org/ , in the category of Fundamental Physics 2016,
https://en.wikipedia.org/wiki/Fundamental_Physics_Prize .

[23] Both Advanced Information for the 2015 and for the 2008 Nobel Prizes in Physics,
https://www.nobelprize.org/nobel_prizes/physics/laureates/2015/advanced-physicsprize2015.pdf and
http://www.nobelprize.org/nobel_prizes/physics/laureates/2008/advanced-physicsprize2008.pdf have used the parameterization for the flavor mixing matrix originally given by L.L. Chau and W.Y. Keung, *Comments on the Parameterization of the Kobayashi-Maskawa Matrix*, Phys. Rev. Lett. **53**, 1802 (1984), http://chau.physics.ucdavis.edu/Chau-Keung-paraKM-PRL53-p1802-1984.pdf ,
which has also been adopted by the Particle Data Group as the "standard choice", http://pdg.lbl.gov/2015/reviews/rpp2014-rev-ckm-matrix.pdf Ref. [3], after an interesting evolution of choices. More importantly, the parameters of the parameterization which is in product form and in terms of exact variables (not approximate) have physical meaning in experiments (such as those honored by the 2015 Nobel Prize in Physics). They are precisely used by Nature to characterize flavor oscillations among quarks and among neutrinos, as well as to characterize CP violations. While CP violation in weak interactions of quark flavor changing has been established (after a long interesting history of theoretical and experimental work), CP violation or no CP violation in neutrino flavor changing is yet to be settled by experiments. It will surely be a milestone in the history of physics when that is settled, and then the parameterization will once again be highlighted.
[Right now data are consistent with three flavors of quarks and three flavors of leptons. However, in the event we are surprised in the future and find more than three flavors, the Chau-Keung type of

parameterizations (in product form and in terms of exact variables) for any N flavors have already been derived in L.L. Chau, *Phase Direct CP Violations and General Mixing Matrices.* Phys. Lett. B 651, 293-297 (2007), http://chau.physics.ucdavis.edu/Chau-Direct-CP-PLb651-p293.pdf .]

Another close encounter with Nobel-Prize-honored discoveries was through the paper by R.F. Peierls, T.L. Trueman, and L.L. (Chau) Wang, *Estimates of Production Cross Sections and Distributions for W Bosons and Hadrons Jets in High Energy p p and p $\bar{p}$ Collisions,* Phys. Rev. D16, 1397 (1977). The results in this paper for the quantities given in the title agreed with those of the 1983 experiments in which the observation of W^+, W^- and Z^0 earned the 1984 Nobel Prize in Physics for C. Rubbia and S. van der Meer, http://www.nobelprize.org/nobel_prizes/physics/laureates/1984/ . After the suspension of several months between the observation of the W's and the observation of Z^0, amazingly (or boringly, depending on one´s point of view) the masses of W^+, W^- and Z^0 were observed at the predicted values based upon the earlier 1979 Nobel-Prize-honored theoretical works by S.L. Glashow, by A. Salam and by S. Weinberg, http://www.nobelprize.org/nobel_prizes/physics/laureates/1979/. This success story concurs once more with Einstein's famous statement, "Subtle is The Lord, but malicious He is not!" (The English translation from German is that of A. Pais, Ref. [5].)

The reason that Peierls, Trueman, and (Chau) Wang did the calculations reported in that paper was because Brookhaven National Laboratory (BNL or Brookhaven) was building the then highest energy *pp* (proton-proton) collider, ISABELLE, in search of W^+, W^- and Z^0. Unfortunately Brookhaven failed to produce magnets with the required properties. This resulted in the cancellation of ISABELLE, after already spending $200M, https://en.wikipedia.org/wiki/ISABELLE (so goes the joke, "IS-A-BELLE became WAS-A-BELLE"). As a consequence, leading experimental physicists left BNL and went to CERN, Geneva, Switzerland to build the proton-anti-proton $p\bar{p}$ collider and then made the observation of W^+, W^- and Z^0. (As the authors of the paper we were told by the experimentalists that the results of substantial $p\bar{p}$ production cross sections for W^+, W^- and Z^0 in our paper gave them confidence.) The momentous event also planted the seed that eventually led to the building of the now world-famed Large Hadron Collider LHC at Geneva, where the discovery of the Higgs particle was made in 2012, Refs. [18,19,20].

Brookhaven had lost such a golden chance. One can only imagine what it would be like for Brookhaven and for US high energy physics had Brookhaven been successful in building ISABELLE.

[24] This is related to the problem of "Yang-Mills and Mass Gap", which is the first of the seven Millennium Problems posted by the Clay Mathematics Institute http://www.claymath.org/millennium-problems , with official problem description at http://www.claymath.org/sites/default/files/yangmills.pdf , and the status as of 2004 at http://www.claymath.org/sites/default/files/ym2.pdf

[25] E. Witten, *The Problem of Gauge Theory*, http://arxiv.org/pdf/0812.4512v3.pdf , where he gave the following concluding statements:

"*The mass gap is the reason, if you will, that we do not see classical nonlinear Yang-Mills waves. They are a good approximation only under inaccessible conditions. I have spent most of my career wishing that we had a really good way to quantitatively understand the mass gap in four-dimensional gauge theory. I hope that this problem will be solved one day*."

See also his, *What We Can Hope To Prove About 3d Yang-Mills Theory*, http://media.scgp.stonybrook.edu/presentations/20120117_3_Witten.pdf
Simons Center, January 17, 2012.

[26] S. Weinberg, Vol. III <*Supersymmetry*>, of Ref. [6]; J. Terning, <*Modern Supersymmetry: Dynamics and Duality*>, (Clarendon Press, Oxford, 2006)

[27] E. Witten, *Quest For Unification*, http://arxiv.org/pdf/hep-ph/0207124.pdf , based on his Heinrich Hertz lecture at SUSY 2002 at DESY, June, 2002.

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http://chau.physics.ucdavis.edu/Text/pdfs/NATO-Intl-Conf-Diff-Geo-Th-Phys-UCD-1988-Jul2-8.pdf

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http://www.shawprize.org/en/shaw.php?tmp=3&twoid=12&threeid=41&fourid=20 , and See him as one of the seven recipients of the inaugural Breakthrough Prizes in Mathematics, 2014, https://en.wikipedia.org/wiki/Breakthrough_Prize_in_Mathematics
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[80] In the opinion of many, including the author, it has been established that mathematics is the precise language of Nature. The author hopes that in some small measure this paper has helped to make the point that everyone, scientists and non-scientists, can participate in the excitement and the enrichment found in the frontiers of scientific discovery, just as everyone can enjoy music and painting without being able to play a musical instrument and to paint. This point of view has guided the author in communicating physics (particle, condensed matter, and cosmology) to non-scientists since the late 1980s. For a sampling of her approach with good results, see the webpage of her 2005 Physics 10 for non-scientist undergraduates, http://www.physics.ucdavis.edu/Classes/Physics10-B-W05/ .