Comments on the Parametrization of the Kobayashi-Maskawa Matrix

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We show that the quark mixing matrix can be parametrized in exactly unitary forms with the imaginary parts present only at the order of 10^{-3}. With $s_x = s_1$ and s_y well determined, measurements of ϵ' or other *CP*-nonconservation effects can determine $s_z s_\phi$. Then after $|V_{ub}| \simeq s_z$ is measured, s_ϕ is known. We also give a simple expression for the *CP* asymmetry in the B^0-$\overline{B}^0$ mixing.

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The recent measurements of the b lifetime[1] have put a very strong constraint on the Kobayashi-Maskawa quark mixing[2] matrix. The salient features are that the weak transitions in the heavier quark sector ($b \to c$ and $t \to s$) are much more suppressed than those in the lighter quark section ($s \to u$ and $c \to d$), e.g.,[3,4]

$$|V_{us}| = 0.23, \quad |V_{cb}| \approx 0.06, \tag{1}$$

where

$$V_{\rm KM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_2 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}. \tag{2}$$

Note that some matrix elements have comparable real and imaginary parts, e.g., V_{tb}. However, we have calculated various violations of *CP* conservation; it is always a small number proportional to $s_2 s_3 s_\delta$. Recently it has been shown by Wolfenstein[5] that by reparametrizing the KM matrix one can see that the imaginary parts in the whole KM matrix appear only with coefficient $\lesssim 10^{-3}$. Nevertheless, the scheme used in Ref. 5 is by a power series expansion, and not exact. Here we first show that the original KM matrix, by redefinition of t and b quark field phases, can be put into an exactly unitary expression so that the imaginary parts only appear in the matrix element of order $\lesssim 10^{-3}$.

We can redefine the phases of the t quark field and the b quark field by $\exp(i\phi_t)$ and $\exp(-i\phi_b)$, respectively, such that the new V_{cb} and V_{ts} become real, i.e.,

$$V_{\rm KM} \to V' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_b} \end{pmatrix} V_{\rm KM} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_b} \end{pmatrix} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 e^{i\phi_b} \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & |c_1 c_2 s_3 + s_2 c_3 e^{i\delta}| \\ -s_1 s_2 e^{i\phi_t} & |c_1 s_2 c_3 + c_2 s_3 e^{i\delta}| & (c_1 s_2 s_3 - c_2 c_3 e^{i\delta}) e^{i\phi_b + i\phi_t} \end{pmatrix}, \tag{3}$$

with

$$e^{i\phi_b} \equiv \frac{c_1 c_2 s_3 + s_2 c_3 e^{-i\delta}}{|c_1 c_2 s_3 + s_2 c_3 e^{i\delta}|}, \quad e^{i\phi_t} \equiv \frac{c_1 s_2 c_3 + c_2 s_3 e^{-i\delta}}{|c_1 s_2 c_3 + c_2 s_3 e^{i\delta}|}. \tag{4}$$

Now let us study to what order of magnitude the imaginary parts begin to appear in the matrix V'. Since s_1, s_2, and s_3 are small, $s_2 \sim s_3 \sim s_1^2 \sim 10^{-2}$, we can approximate $c_1 \approx 1 - s_1^2/2 + O(10^{-4})$, $c_2 \sim c_3 \approx 1 + O(10^{-4})$. Then Eq. (3) becomes

$$V' \sim \begin{pmatrix} 1 - s_1^2/2 & s_1 & \dfrac{s_1 s_3 (s_3 + s_2 e^{-i\delta})}{(s_3^2 + s_2^2 + 2 s_2 s_3 c_\delta)^{1/2}} \\ -s_1 & 1 - s_1^2/2 - s_2 s_3 e^{i\delta} & (s_3^2 + s_2^2 + 2 s_2 s_3 c_\delta)^{1/2} \\ \dfrac{-s_1 s_2 (s_2 + s_3 e^{-i\delta})}{(s_2^2 + s_3^2 + 2 s_2 s_3 c_\delta)^{1/2}} & (s_2^2 + s_3^2 + 2 s_2 s_3 c_\delta)^{1/2} & -1 \end{pmatrix}. \tag{5}$$

We note that the transformed matrix elements are all real up to order $O(10^{-3})$. This indicates that although the original KM matrix elements can have large imaginary parts, they do not give appreciable *CP*-

nonconservation effects. Here we have shown the point made by Wolfenstein in an exactly unitary form.

Though V' contains the points made by Wolfenstein, it is still not in a neat form. After some trials we find another way to parametrize the 3×3 unitary KM matrix, i.e.,

$$V=\begin{pmatrix}1 & 0 & 0\\ 0 & c_y & s_y\\ 0 & -s_y & c_y\end{pmatrix}\begin{pmatrix}c_z & 0 & s_z e^{-i\phi}\\ 0 & 1 & 0\\ -s_z e^{i\phi} & 0 & c_z\end{pmatrix}\begin{pmatrix}c_x & s_x & 0\\ -s_x & c_x & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}c_x c_z & s_x c_z & s_z e^{-i\phi}\\ -s_x c_y - c_x s_y s_z e^{i\phi} & c_x c_y - s_x s_y s_z e^{i\phi} & s_y c_z\\ s_x s_y - c_x c_y s_z e^{i\phi} & -c_x s_y - s_x c_y s_z e^{i\phi} & c_y c_z\end{pmatrix}. \quad (6)$$

Now from the measurements of the b lifetime and $|V_{ub}/V_{cb}|\leqslant 0.14$, and from Ref. 4, we have

$$|V_{ub}|=s_z\leqslant 6.96\times 10^{-3},\ 8.24\times 10^{-3},\ 1.06\times 10^{-2}, \quad (7)$$

for $\tau_b=1.4$, 1.0, 0.6 psec, respectively. *Since s_z is very small and of order 10^{-3}*, then we have

$$|V_{cb}|=|s_y c_z|=s_y[1+O(10^{-4})] =0.050,\ 0.059,\ 0.076, \quad (8)$$

for $\tau_b=1.4$, 1.0, 0.6 psec, respectively. *Also s_y is a rather small number of order* 10^{-2}, in comparison with s_x *(of order 10^{-1})* which is obtained from

$$|V_{us}|=s_x[1+O(10^{-4})]=0.23, \quad (9)$$

via hyperon and K_{e3} decays; and

$$|V_{ud}|=c_x[1+O(10^{-4})]=0.973, \quad (10)$$

via $0^+\to 0^+$ nuclear β decay. Now,

$$V=\begin{pmatrix}c_x & s_x & s_z e^{-i\phi}\\ -s_x - s_y s_z e^{i\phi} & c_x & s_y\\ s_x s_y - s_z e^{i\phi} & -s_y - s_x s_z e^{i\phi} & 1\end{pmatrix}. \quad (11)$$

We keep the real parts and the imaginary parts accurate up to 10^{-4} and 10^{-6}, respectively. It is clear that the CP-nonconservation imaginary parts appear only with coefficients $\lesssim 10^{-3}$ in the whole matrix.

The s_x, s_y, s_z, and s_ϕ are related to the original s_1, s_2, s_3, and s_δ of the KM matrix by

$$s_x=s_1+O(10^{-4}),$$
$$s_y=(s_2^2+s_3^2+2s_2s_3c_\delta)^{1/2}+O(10^{-4}), \quad (12)$$
$$s_z=s_1s_3,\quad s_y s_\phi=s_2 s_\delta+O(10^{-4}).$$

These relations are graphically illustrated in Fig. 1.

It turns out that our parametrization is very close to the Maiani parametrization.[6] They differ by a phase transformation to the t and b quark fields, i.e.,

$$V_{\text{Maiani}}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-i\phi}\end{pmatrix}V\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{+i\phi}\end{pmatrix}=\begin{pmatrix}c_x c_z & s_x c_z & s_z\\ -s_x c_y - c_x s_y s_z e^{i\phi} & c_x c_y - s_x s_y s_z e^{i\phi} & s_y c_z e^{i\phi}\\ -c_x c_y s_z + s_x s_y e^{-i\phi} & -s_x c_y s_z - c_x s_y e^{-i\phi} & c_y c_z\end{pmatrix}. \quad (13)$$

The only disadvantage to using Maiani's phase convention is that the imaginary parts in V_{cb} and V_{ts} have a coefficient of 10^{-2}.

Now we discuss some physical implications on these new parameters s_x, s_y, s_z, and s_ϕ. It is interesting to note that the limits on the angles from the b lifetime alone are already more stringent than those from hyperon decays, $s_z<0.2$. The current bound is $s_z<0.14 s_y\lesssim 10^{-2}$. Also these bounds provide a stronger limit on $|V_{td}V_{ts}|$,

$$|V_{dt}|\leqslant s_x s_y+s_z\leqslant 2 s_x s_y, \quad (14)$$

$$|V_{td}V_{ts}|\leqslant 2 s_x s_y s_y=1.15\times 10^{-3},\ 1.6\times 10^{-3},\ 2.6\times 10^{-3}, \quad (15)$$

for $\tau_b=1.4$, 1.0, 0.6 psec, respectively, than previously obtained from $K_L\to\mu^+\mu^-$ which gives $|\text{Re}(V_{td}V_{ts})|<0.02$, for $m_t\sim 40$ GeV based on the

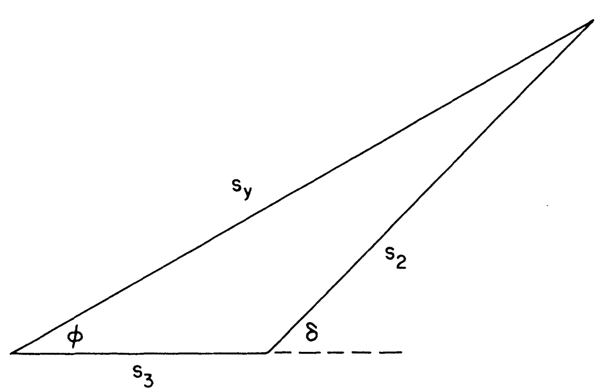

FIG. 1. Graphical relations between the conventional parametrization and our proposed parametrization.

box diagram calculation.

It is also of interest to note that all *CP*-nonconservation effects of *first order in weak interactions,* be it[7] ϵ' or the partial-decay-rate difference between the charged[8] s, c, b, or t decays and their *CP* conjugate processes, are proportional to a universal factor X_{CP} accurate up to 10^{-8} from the quark mixing matrix,

$$X_{CP} = s_1^2 s_2 s_3 s_\delta = s_x s_y s_z s_\phi. \tag{16}$$

(Note that the *CP* mixing parameter ϵ is a result of second-order weak interaction.) For the strange and charm decays, the *CP*-nonconservation factors from the quark mixing matrix are $\mathrm{Im}[(V_{us}V_{ud}^*)\times(V_{cs}V_{cd}^*)^*]$ and $\mathrm{Im}[(V_{us}V_{cs}^*)(V_{ud}V_{cd}^*)^*]$, respectively. For the b decays, they are $\mathrm{Im}[(V_{cb}V_{cs}^*)\times(V_{ub}V_{us}^*)^*]$ or $\mathrm{Im}[(V_{cb}V_{cd}^*)(V_{ub}V_{ud}^*)^*]$, and for the t decays they are $\mathrm{Im}[(V_{ti}V_{ki}^*)(V_{tj}V_{kj}^*)^*]_{i\neq j}$, where $i,j = d$, s, or b, and $k = u$ or c. But it can be easily shown, especially in our proposed parametrization, that they are all equal to X_{CP} with corrections of order 10^{-8}. This implies that in the study of *CP* nonconservation we can never separate $s_3 s_\delta$ or $s_z s_\phi$, so that s_3 or s_z need to be obtained from other reactions. The nicest one is from the study of $|V_{ub}| = s_z = s_1 s_3$. A candidate reaction is $B_u^- \to \tau^-\nu_\tau$ (here we choose $\tau^-\nu_\tau$ rather than $\mu^-\nu_\mu$ to avoid suppressions from helicity conservation). Unfortunately the branching fraction[9] of such decay is very small, $B_r(B_u \to \tau\nu_\tau) \lesssim 7.4\times10^{-5}$.

In Fig. 2 we give the value of $X_{CP}/s_1^2 = s_2 s_3 s_\delta = s_y s_z s_\phi / s_x$ determined from the b lifetime and the fitting of ϵ via the box diagram calculation for various m_t. The measurement of $\epsilon' = 0.02 s_2 s_3 s_\delta = 0.02 s_y s_z s_\phi / s_x$ will give us the value of $s_z s_\phi$ [since s_x and s_y are known, Eqs. (8) and (9), but not $s_3 s_\delta$ since s_2 is not yet determined independently; here we see the advantage of using the s_x, s_y, s_z, s_ϕ parametrization].

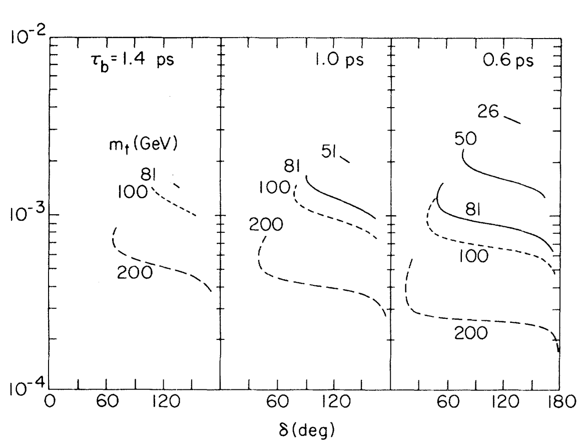

FIG. 2. The size of $X_{CP}/s_1^2 = s_2 s_3 s_\delta = s_y s_z s_\phi / s_x$ vs δ for $\tau_b = 0.6$, 1.0, and 1.4 psec for various m_t.

The *CP* nonconservation could also appear significantly in the B^0-$\bar{B}^0$ mixing.[4,10,11] It is characterized by an asymmetry parameter

$$a(B^0) = P^-/P^+, \tag{17a}$$

where the time-integrated probability is

$$P^\pm = p(\bar{B}^0 \to B^0) \pm p(B^0 \to \bar{B}^0), \tag{17b}$$

with p denoting "probability." For the neutral mesons $B_s(b\bar{s})$ and $B_d(b\bar{d})$, we obtain simple expressions for $a(B^0)$ in the context of the standard model,

$$a(B_s) = -\frac{4\pi}{F(m_t^2/m_W^2)} \frac{m_c^2}{m_t^2} \frac{c_x s_x s_z s_\phi}{s_y},$$
$$a(B_d) = -a(B_s) s_y^2 / (c_x |s_x s_y - s_z e^{i\phi}|^2), \tag{18}$$

with

$$F(x) \equiv 1 - 0.75\frac{(x+x^2)}{(1-x)^2} - 1.5\frac{x^2 \ln x}{(1-x)^3}.$$

Note that for neutral mesons our theorem Eq. (16) only applies to the moment, say $t=0$, when the particle is a definite P^0 or $\bar{P}^0$, where $P^0(\bar{P}^0)$ denotes neutral mesons of any flavor, e.g., $K^0(\bar{K}^0)$, $D^0(\bar{D}^0)$, $B^0(\bar{B}^0)$. At later moments $P^0(\bar{P}^0)$ mix through second-order weak interactions in the KM scheme and become mixed states of P^0 and $\bar{P}^0$, in which case our theorem does not apply. The phenomena pointed out in Ref. 12 belong to this category.

The calculation, Eq. (18), ignores QCD corrections. We also drop terms of order $(m_c/m_b)^4$ in Γ_{12} or $(m_c/m_t)^2$ in M_{12}, which are verified numerically to be negligible. The ranges of *CP* nonconservations[4] are

$$a(B_d) \sim 10^{-2}\text{–}10^{-3},$$
$$-a(B_s) \sim 0.5\times10^{-3}\text{–}0.5\times10^{-4}, \tag{19}$$

for $m_t \lesssim 100$ GeV. The *CP*-nonconservation effects could give rise to the observable difference in event rates between the $\mu^+\mu^+$ and $\mu^-\mu^-$ dimuons originating from $b\bar{b}$ production in high-energy collider experiments.

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